Exam 2 Study Guide

Math 290 students are responsible for the following topics, highlighted in blue, that will be covered in Exam 2. The format of Exam 2 will be similar to that of Exam 1, namely, the exam will be comprised of True-False and short answer questions and problems that require solving equations or calculating quantities associated with various matrices. Studying highlighted statements or statements deemed to be important in the class slides is a good way to prepare for the True-False and short answer questions.

0. Calculating determinants. Knowing methods for calculating determinants is important for the topics that follow.

1. Eigenvalues, eigenvectors, and diagonalizability of square matrices. Let A be an $n \times n$ matrix.

(i) The real number λ is an **eigenvalue** of A is there exists a **non-zero** vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. In this case, v is an **eigenvector** associate to λ.

(ii) The eigenvalues of A are the roots of $c_A(x)$, the *characteristic polynomial* of A. $c_A(x) = \det[xI_n - A]$.

(iii) For a given eigenvalue λ, the λ-eigenvectors are the non-zero vectors in the null space of the matrix $\lambda I_n - A$. The basic solutions in this null space are **basic** λ-eigenvectors and form a **basis** for the eigenspace E_λ.

(iv) If A is an $n \times n$ matrix, then, by definition, A is **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$, where D is an $n \times n$ diagonal matrix.

(v) If A is diagonalizable, the diagonal entries of the matrix D in (iv) are the eigenvalues of A.

(vi) Suppose $c_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$, then the eigenvalue λ_i has **multiplicity** e_i.

(vii) A is diagonalizable if and only if $c_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ and for each eigenvalue λ_i, e_i equals the dimension of E_{λ_i}.

(viii) If A is diagonalizable, then the diagonalizing matrix P is obtained by taking the matrix whose columns are the collection of basic eigenvectors derived from A.

2. Applications of diagonalizability of square matrices. Suppose A is diagonalizable, with $P^{-1}AP = D$, a diagonal matrix.

(i) $A = PDP^{-1}$, and therefore $A^n = PD^nP^{-1}$, for all $n \geq 1$.

(ii) For any square matrix B, e^B is the matrix given by the Taylor Series: $\sum_{n=0}^{\infty} \frac{1}{n!} B^n$.

(iii) If $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$, then $e^D = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.

(iv) For A diagonalizable, $e^A = Pe^DP^{-1}$.

(v) Solving recurrence relations: A sequence of non-negative numbers $a_0, a_1, a_2, \ldots, a_k, \ldots$, is called a **linear recursion sequence of length two** if there are fixed integers α, β, c, d such that:

(i) $a_0 = \alpha$.

(ii) $a_1 = \beta$.

(iii) $a_{k+2} = c \cdot a_k + d \cdot a_{k+1}$, for all $k \geq 0$.

To find a closed form solution for a_k, let $v_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$, and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$. Then $v_k = A^k \cdot v_0$, and a_k is the first coordinate of the vector v_k.

(vi) Solving systems of first order linear differential equations: Let $A = (a_{ij})$, be an $n \times n$ matrix. A system of first order linear differential equations is a system of equations of the form:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + \cdots + a_{2n}x_n(t) \\ \vdots &= \qquad \vdots \\ x_n'(t) &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t), \end{aligned}$$

where $x_i(t)$ is a real valued function of t. The numbers $x_1(0), \cdots, x_n(0)$ are called the *initial conditions* of the system. In matrix form, the system is given by the equation: $\mathbf{X}'(t) = A \cdot \mathbf{X}(t)$. **The solution to the system is given by:** $\mathbf{X}(t) = e^{At} \cdot \mathbf{X}(0)$.

3. Spanning sets, linear independence and bases in Euclidean space. Let $v_1, \dots, v_r, w$ be columns vectors in $\mathbb{R}^n$. Let $A = [v_1\ v_2 \cdots\ v_r]$, the matrix whose columns are $v_1, \dots, v_r$. Then:

(i) w belongs to $\text{span}\{v_1, \dots, v_r\}$ if and only if the system of equations $A \cdot \mathbf{X} = w$ has a solution.

(ii) If $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a solution to $A \cdot \mathbf{X} = w$, then $w = \lambda_1 v_1 + \cdots + \lambda_r v_r$.

(iii) $v_1, \dots, v_r$ are linearly independent if and only if $A \cdot \mathbf{X} = \mathbf{0}$ has only the zero solution.

(iv) If $v_1, \dots, v_r$ are not linearly independent and $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a non-zero solution to $A \cdot \mathbf{X} = \mathbf{0}$, then

$$(*) \quad \lambda_1 v_1 + \cdots \lambda_r v_r = \mathbf{0}.$$

This means the vectors $v_1, \dots, v_r$ are linearly dependent, and thus redundant.

(v) One can use (*) to write some v_i in terms of the remaining v's. Upon doing so:

$$\text{span}\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_r\} = \text{span}\{v_1, \dots, v_r\}.$$

(vi) One may continue to eliminate redundant vectors from among the v_i's. As soon as one one arrives at a linearly independent subset of $v_1, \dots, v_r$, this set of vectors forms a basis for the original subspace $\text{span}\{v_1, \dots, v_r\}$. The number of elements in the basis is then the dimension of $\text{span}\{v_1, \dots, v_r\}$.

(vii) To test if the n vectors $v_1, \dots, v_n$ in $\mathbb{R}^n$ are linearly independent, or span $\mathbb{R}^n$, or form a basis for $\mathbb{R}^n$, it suffices to show that $\det[v_1\ v_2\ \cdots\ v_n] \neq 0$.